DREAMMONGERI EDUCATION

AP® CALCULUS AB & BC — UNIT 1

Limits and Continuity · Formula & Reference Sheet

LIMIT NOTATION AND WHEN A LIMIT EXISTS

Two-sided limit

$\lim_{x \to a} f(x) = L$ means $f(x)$ gets as close to L as we like for every x close enough to a on both sides, $x \neq a$.

The value $f(a)$ — or whether $f(a)$ exists — does not affect the limit.

One-sided limits

Left: $\lim_{x \to a^-} f(x)$ Right: $\lim_{x \to a^+} f(x)$

Existence

$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

A limit does **not** exist when the one-sided limits differ, when f grows without bound, or when f oscillates without settling.

Writing $\lim f(x) = \infty$ describes unbounded behavior; the limit still does not exist as a number.

From graphs and tables

Read the y-values approached from each side. A table suggests a limit; it does not prove one.

LIMIT LAWS

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ both exist:

Law	Result	Law	Result
Sum / difference	$\lim [f \pm g] = L \pm M$	Product	$\lim [fg] = LM$
Constant multiple	$\lim [kf] = kL$	Quotient	$\lim \dfrac{f}{g} = \dfrac{L}{M}$ $(M \neq 0)$
Power	$\lim [f]^n = L^n$	Root	$\lim \sqrt[n]{f} = \sqrt[n]{L}$ $(L > 0$ if n even$)$
Composition	$\lim_{x \to a} h(g(x)) = h(M)$ when h is continuous at M		
Direct substitution	$\lim_{x \to a} f(x) = f(a)$ whenever f is continuous at a — polynomials everywhere; rational, trig, exponential and log functions at points of their domains		

ALGEBRAIC TECHNIQUES FOR THE FORM $\frac{0}{0}$

1. Factor and cancel

$a^2 - b^2 = (a - b)(a + b)$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

2. Multiply by the conjugate

$(\sqrt{A} - \sqrt{B})(\sqrt{A} + \sqrt{B}) = A - B$

3. Combine fractions

Rewrite over a common denominator, then simplify.

4. Absolute value — split by side

$|x - a| = x - a$ for $x > a$; $|x - a| = a - x$ for $x < a$. Find each one-sided limit.

⚠ $\frac{0}{0}$ means "more work needed" — not 0, not DNE.

⚠ $\frac{k}{0}$ with $k \neq 0$: unbounded; check the sign on each side.

SPECIAL TRIGONOMETRIC LIMITS (x IN RADIANS)

Limit	Value	Limit	Value
$\lim_{x \to 0} \dfrac{\sin x}{x}$	1	$\lim_{x \to 0} \dfrac{1 - \cos x}{x}$	0
$\lim_{x \to 0} \dfrac{\sin(kx)}{x}$	k	$\lim_{x \to 0} \dfrac{\tan x}{x}$	1
$\lim_{x \to 0} \dfrac{\sin(ax)}{\sin(bx)}$ $(b \neq 0)$	$\dfrac{a}{b}$	$\lim_{x \to 0} \dfrac{x}{\sin x}$	1

SQUEEZE THEOREM

If $g(x) \leq f(x) \leq h(x)$ for all $x \neq a$ near a, and $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} f(x) = L$.

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Example: $-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$ for $x \neq 0$, and both bounds $\to 0$, so $\lim\limits_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$.

CONTINUITY

Continuous at $x = a$ — all three must hold

1. $f(a)$ is defined
2. $\lim\limits_{x\to a} f(x)$ exists
3. $\lim\limits_{x\to a} f(x) = f(a)$

Continuous on $[a, b]$

Continuous on (a, b), $\lim\limits_{x\to a^+} f(x) = f(a)$ and $\lim\limits_{x\to b^-} f(x) = f(b)$.

Continuous on their domains

Polynomials, rational, power and root functions, trig and inverse trig, exponential and logarithmic functions.

Sums, differences, products, quotients (denominator $\neq 0$) and compositions of continuous functions are continuous.

Removing a discontinuity

If $\lim\limits_{x\to a} f(x) = L$ exists, define (or redefine) $f(a) = L$.

Piecewise, at a boundary c: require $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x) = f(c)$.

Types of discontinuity

Type	Condition at $x = a$	Graph
Removable	$\lim_{x\to a} f(x)$ exists, but $f(a)$ is undefined or $f(a) \neq \lim_{x\to a} f(x)$	Hole (point may be moved)
Jump	Both one-sided limits exist (finite) but are not equal	Graph jumps between two values
Infinite	At least one one-sided limit is $+\infty$ or $-\infty$	Vertical asymptote at $x = a$

ASYMPTOTES AND LIMITS AT INFINITY

Vertical asymptote $x = a$

$\lim\limits_{x\to a^-} f(x)$ or $\lim\limits_{x\to a^+} f(x)$ is $\pm\infty$.

For a rational function: zeros of the denominator that do not cancel. A factor that cancels gives a hole instead.

Horizontal asymptote $y = L$

$\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$. Check the two directions separately — they can differ.

A graph may cross its horizontal asymptote (e.g. $\frac{\sin x}{x}$ crosses $y = 0$).

Rational functions $\dfrac{p(x)}{q(x)}$, degrees m and n

- $m < n$: horizontal asymptote $y = 0$
- $m = n$: $y = \dfrac{\text{leading coeff. of } p}{\text{leading coeff. of } q}$
- $m > n$: no horizontal asymptote

Limits worth knowing

$\lim_{x\to\infty} e^x = \infty$	$\lim_{x\to-\infty} e^x = 0$
$\lim_{x\to\infty} \ln x = \infty$	$\lim_{x\to 0^+} \ln x = -\infty$
$\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$	$\lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}$
$\lim_{x\to\infty} \dfrac{1}{x^p} = 0$ for every $p > 0$	

As $x \to \infty$: $\ln x$ grows slower than x^p $(p > 0)$, which grows slower than e^x.

INTERMEDIATE VALUE THEOREM (IVT)

If f is **continuous on** $[a, b]$ and N is strictly between $f(a)$ and $f(b)$, then there is at least one c in (a, b) with $f(c) = N$.

Root version: continuous on $[a, b]$ and $f(a)$, $f(b)$ of opposite signs $\Rightarrow$ $f(c) = 0$ for some c in (a, b). IVT gives existence only, not how many.

WORKED EXAMPLES

Example. $\lim\limits_{x\to 3} \dfrac{x^2 - 9}{x - 3}$

$\dfrac{(x-3)(x+3)}{x-3} = x + 3$ for $x \neq 3$, so the limit is $3 + 3 = 6$.

Example. $\lim\limits_{x\to\infty} \dfrac{3x^2 - x}{5x^2 + 2}$ **and** $\lim\limits_{x\to-\infty} \dfrac{2x + 1}{\sqrt{x^2 + 1}}$

Equal degrees: $\frac{3}{5}$. For $x < 0$, $\sqrt{x^2} = |x| = -x$, so divide by $-x$: $\dfrac{-2 - 1/x}{\sqrt{1 + 1/x^2}} \to -2$.

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Example. $\lim_{x \to 0} \dfrac{\sqrt{x+4}-2}{x}$

Multiply by $\sqrt{x+4}+2$:

$$\frac{x}{x(\sqrt{x+4}+2)} = \frac{1}{\sqrt{x+4}+2} \to \frac{1}{4}.$$

Example. $\lim_{x \to 0} \dfrac{\sin(5x)}{3x}$

$$\frac{5}{3} \cdot \frac{\sin(5x)}{5x} \to \frac{5}{3} \cdot 1 = \frac{5}{3}.$$

Example. Find k so that $f(x) = \begin{cases} kx+1, & x < 2 \\ x^2 - k, & x \geq 2 \end{cases}$ is continuous.

Left limit $2k+1$, right limit $= f(2) = 4 - k$.

$2k + 1 = 4 - k \Rightarrow k = 1$.

Example. Show $x^3 + x - 1 = 0$ has a solution in $(0, 1)$.

$f(x) = x^3 + x - 1$ is a polynomial, so continuous on $[0, 1]$. $f(0) = -1 < 0$, $f(1) = 1 > 0$. By the IVT, $f(c) = 0$ for some c in $(0, 1)$.

COMMON ERRORS

- Using direct substitution where f is not continuous (e.g. at a piecewise boundary).
- Treating $\frac{0}{0}$ as an answer instead of a signal to simplify.
- Assuming the limits as $x \to \infty$ and $x \to -\infty$ are the same.
- Applying the IVT without stating that f is continuous on the closed interval.

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