AP® CALCULUS BC — UNIT 10

Infinite Sequences and Series · Formula & Reference Sheet

BC ONLY This whole unit is tested on the BC exam only.

SERIES: CONVERGENCE AND DIVERGENCE

Partial sums: $S_n = a_1 + a_2 + \cdots + a_n$. $\sum_{n=1}^{\infty} a_n$ **converges** to S if $\lim_{n\to\infty} S_n = S$ (finite); otherwise it **diverges**.

Adding or removing finitely many terms can change the sum, but not whether the series converges.

Geometric series

$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots = \frac{a}{1-r}$ if $|r| < 1$; diverges if $|r| \geq 1$ ($a \neq 0$).

a = first term, r = ratio of consecutive terms.

CONVERGENCE TESTS

Test	Statement	Notes
nth-term test	If $\lim a_n \neq 0$ (or does not exist), $\sum a_n$ diverges.	$\lim a_n = 0$ proves nothing.
Integral test	f positive, continuous, decreasing for $x \geq N$, $a_n = f(n)$: $\sum a_n$ and $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.	The sum is not equal to the integral.
p-series	$\sum \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \leq 1$.	$p = 1$: harmonic series, diverges.
Direct comparison	$0 \leq a_n \leq b_n$: $\sum b_n$ converges $\Rightarrow \sum a_n$ converges; $\sum a_n$ diverges $\Rightarrow \sum b_n$ diverges.	Compare with geometric or p-series.
Limit comparison	$a_n, b_n > 0$, $\lim \frac{a_n}{b_n} = c$, $0 < c < \infty$: both converge or both diverge.	Take b_n = dominant terms of a_n.
Alternating series test	$\sum(-1)^n b_n$, $b_n > 0$: if b_n is decreasing and $\lim b_n = 0$, the series converges.	Shows convergence only — not absolute.
Ratio test	$L = \lim \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$: $L < 1$ converges absolutely; $L > 1$ diverges; $L = 1$ inconclusive.	Best with factorials and powers.

Absolute vs. conditional convergence

Absolutely convergent: $\sum |a_n|$ converges — then $\sum a_n$ converges too.

Conditionally convergent: $\sum a_n$ converges but $\sum |a_n|$ diverges, e.g. $\sum \frac{(-1)^{n+1}}{n}$.

Alternating series error bound

If $\sum(-1)^n b_n$ satisfies the alternating series test with sum S, then $|S - S_n| \leq b_{n+1}$, the first omitted term.

S lies between consecutive partial sums.

TAYLOR POLYNOMIALS AND THE LAGRANGE ERROR BOUND

Taylor polynomial of degree n about $x = a$:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

P_n and f agree in value and first n derivatives at a.

Maclaurin: $a = 0$. Coefficient of $(x-a)^k$ is $\frac{f^{(k)}(a)}{k!}$.

Lagrange error bound: if $|f^{(n+1)}(z)| \leq M$ for all z between a and x, then

$$|f(x) - P_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$

For $\sin$ and $\cos$, $M = 1$ works. For an alternating series with decreasing terms, the alternating series bound may be used instead.

POWER SERIES: RADIUS AND INTERVAL OF CONVERGENCE

$\sum_{n=0}^{\infty} c_n(x-a)^n$ does exactly one of: converges only at $x = a$ ($R = 0$); converges for all x ($R = \infty$); or converges for $|x-a| < R$ and diverges for $|x-a| > R$.

Interval: test both endpoints $x = a \pm R$ separately (p-series, alternating series, nth-term tests).

Differentiating or integrating term by term keeps the same radius; endpoint behavior can change.

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Find R with the ratio test: solve $\lim \left|\frac{c_{n+1}}{c_n}\right| |x-a| < 1$.

MACLAURIN SERIES TO KNOW

$f(x)$	Series	Converges for
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	all x
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	all x
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	all x
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$	$-1 < x \le 1$
$\arctan x$	$x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$	$-1 \le x \le 1$

The last two come from $\frac{1}{1-x}$: substitute $-x$ (or $-x^2$), then integrate term by term. New series from known ones: substitute (e.g. x^2 for x), multiply by a power of x, differentiate, or integrate.

WORKED EXAMPLES

Example. $\sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n$ **and** $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$

$\frac{3}{1-1/2} = 6$; first term $\frac{2}{3}$: $\frac{2/3}{1-2/3} = 2$.

Example. Interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{n\,3^n}$.

Ratio: $\frac{n}{n+1} \cdot \frac{|x|}{3} \to \frac{|x|}{3} < 1 \Rightarrow R = 3$. $x = 3$: $\sum \frac{1}{n}$ diverges. $x = -3$: $\sum \frac{(-1)^n}{n}$ converges (AST). Interval $[-3, 3)$.

Example. Error bound for $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ **using** S_3.

$S_3 = 1 - \frac{1}{4} + \frac{1}{9} = \frac{31}{36} \approx 0.8611$.
$|S - S_3| \le b_4 = \frac{1}{16} = 0.0625$. (Actual $S = \frac{\pi^2}{12} \approx 0.8225$.)

Example. Convergence of $\sum \frac{2n+3}{n^3+5}$ **and** $\sum \frac{(-1)^n}{\sqrt{n}}$.

Limit comparison with $\frac{1}{n^2}$: ratio $\to 2$, so it converges.
$\sum \frac{(-1)^n}{\sqrt{n}}$ converges by AST, but $\sum \frac{1}{\sqrt{n}}$ diverges ($p = \frac{1}{2}$): conditionally convergent.

Example. Third-degree Taylor polynomial for $\ln x$ **about** $x = 1$.

$f(1) = 0,\ f'(1) = 1,\ f''(1) = -1,\ f'''(1) = 2$:
$P_3 = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$.

Example. Bound the error of $P_3(x) = x - \frac{x^3}{6}$ **for** $\sin(0.1)$.

$|f^{(4)}| = |\sin z| \le 1$: error $\le \frac{(0.1)^4}{4!} \approx 4.17 \times 10^{-6}$.
$P_3(0.1) \approx 0.0998333$.

Example. Maclaurin series for e^{-x^2} **and** $x\cos x$.

$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2} - \cdots$;
$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!} = x - \frac{x^3}{2} + \frac{x^5}{24} - \cdots$.

Example. $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ **and** $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ **(ratio test).**

$\frac{n+1}{2} \to \infty$: diverges. $\frac{2}{n+1} \to 0 < 1$: converges (to $e^2 - 1$).

COMMON ERRORS

- Concluding convergence from $\lim a_n = 0$.
- Forgetting to test the endpoints of the interval of convergence.
- Lagrange bound: using $f^{(n)}$ instead of $f^{(n+1)}$, or $n!$ instead of $(n+1)!$.
- Writing the sum of a geometric series that starts at $n = 1$ with a = the $n = 0$ term.

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