AP® CALCULUS AB & BC — UNIT 2

Differentiation: Definition and Fundamental Properties · Formula & Reference Sheet

AVERAGE AND INSTANTANEOUS RATE OF CHANGE

Average rate of change on $[a, b]$

$\frac{f(b) - f(a)}{b - a}$ = slope of the secant line through $(a, f(a))$ and $(b, f(b))$.

Instantaneous rate of change at $x = a$

$f'(a)$ = slope of the tangent line at $(a, f(a))$.

Units

Units of $f'(x)$ = (units of f) per (unit of x). Example: f in meters, x in seconds $\Rightarrow$ f' in meters per second.

THE DERIVATIVE: DEFINITION AND NOTATION

At a point

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

or, equivalently, $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

As a function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Notation

$f'(x)$, y', $\frac{dy}{dx}$, $\frac{d}{dx}[f(x)]$. At a point: $f'(a)$ or $\left.\frac{dy}{dx}\right|_{x=a}$.

Recognizing a derivative in a limit

$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ is $f'(a)$: identify f and a, then differentiate instead of expanding.

Tangent line at $x = a$

$$y - f(a) = f'(a)(x - a)$$

Estimating $f'(a)$ from data

Use the nearest points that bracket a, with $x_1 < a < x_2$:

$$f'(a) \approx \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Symmetric form: $f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}$

DIFFERENTIABILITY AND CONTINUITY

The one-way implication

f differentiable at $a \implies f$ continuous at a.

The converse is **false**: continuous does not imply differentiable. Example: $f(x) = |x|$ is continuous at 0 but not differentiable there.

Equivalent form: not continuous at $a \Rightarrow$ not differentiable at a.

Where a continuous f fails to be differentiable

- **Corner:** one-sided derivatives exist but differ (e.g. $|x|$ at 0: left -1, right 1).
- **Cusp:** the slopes approach $+\infty$ on one side and $-\infty$ on the other (e.g. $x^{2/3}$ at 0).
- **Vertical tangent:** the slopes approach $+\infty$ on both sides, or $-\infty$ on both sides (e.g. $x^{1/3}$ at 0).

⚠ "$f'(a)$ does not exist" does not mean "vertical tangent": $|x|$ at 0 is a corner.

BASIC DERIVATIVE RULES

Rule	Formula	Rule	Formula
Constant	$\frac{d}{dx}[c] = 0$	Power (any real n)	$\frac{d}{dx}[x^n] = nx^{n-1}$
Constant multiple	$\frac{d}{dx}[cf] = cf'$	Sum / difference	$\frac{d}{dx}[f \pm g] = f' \pm g'$
Product	$\frac{d}{dx}[fg] = f'g + fg'$	Quotient	$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{f'g - fg'}{g^2}$

Power-rule rewrites: $\sqrt{x} = x^{1/2} \Rightarrow \frac{1}{2\sqrt{x}}$; $\frac{1}{x} = x^{-1} \Rightarrow -\frac{1}{x^2}$; $\frac{1}{x^3} = x^{-3} \Rightarrow -\frac{3}{x^4}$.

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DERIVATIVES OF ELEMENTARY FUNCTIONS

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$	$\csc x$	$-\csc x \cot x$
e^x	e^x	$\ln x$	$\frac{1}{x}\ (x>0)$

Trig derivatives assume x in radians. $\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}$, $\sec x = \frac{1}{\cos x}$, $\csc x = \frac{1}{\sin x}$ *— the* tan, cot, sec *and* csc *rows follow from these and the quotient rule.*

WORKED EXAMPLES

Example. Use the definition to find $f'(3)$ for $f(x) = x^2$.

$\frac{(3+h)^2 - 9}{h} = \frac{6h + h^2}{h} = 6 + h \to 6$. So $f'(3) = 6$.

Example. Evaluate $\lim_{h\to 0} \frac{(2+h)^3 - 8}{h}$.

This is $f'(2)$ for $f(x) = x^3$: $f'(x) = 3x^2$, so the limit is 12.

Example. Evaluate $\lim_{x\to \pi} \frac{\cos x + 1}{x - \pi}$.

This is $f'(\pi)$ for $f(x) = \cos x$ (since $\cos \pi = -1$): $-\sin \pi = 0$.

Example. Differentiate $y = x^2 \sin x$ and evaluate at $x = \pi$.

$y' = 2x \sin x + x^2 \cos x$. At π: $0 + \pi^2(-1) = -\pi^2$.

Example. Differentiate $y = \frac{x}{x^2+1}$ and evaluate at $x = 2$.

$y' = \frac{(1)(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1 - x^2}{(x^2+1)^2}$. At 2: $-\frac{3}{25}$.

Example. Tangent line to $y = e^x + \ln x$ at $x = 1$.

$y(1) = e$, $y' = e^x + \frac{1}{x} \Rightarrow y'(1) = e + 1$. Line: $y - e = (e+1)(x-1)$.

COMMON ERRORS

- $(fg)' \neq f'g'$ and $\left(\frac{f}{g}\right)' \neq \frac{f'}{g'}$ — use the product and quotient rules.
- Reversing the quotient-rule numerator: it is $f'g - fg'$, in that order.
- Dropping the minus sign on the derivatives of $\cos x$, $\cot x$, $\csc x$.
- Concluding "differentiable" from "continuous", or "vertical tangent" from "f' undefined".

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