AP® CALCULUS AB & BC — UNIT 3

Differentiation: Composite, Implicit, and Inverse Functions · Formula & Reference Sheet

THE CHAIN RULE

Statement

$\frac{d}{dx}\big[f(g(x))\big] = f'(g(x)) \cdot g'(x)$

Leibniz form: if $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Derivative of the outside, evaluated at the inside, times the derivative of the inside. Nested: work from the outside in, one layer at a time.

From a table

If $h(x) = f(g(x))$: $h'(a) = f'(g(a)) \cdot g'(a)$. Look up $g(a)$ first, then f' at that value.

$\frac{d}{dx}$ of	Result ($u = u(x)$)	$\frac{d}{dx}$ of	Result
u^n	$nu^{n-1}\,u'$	$\sqrt{u}$	$\dfrac{u'}{2\sqrt{u}}$
$\sin u$	$\cos u \cdot u'$	$\cos u$	$-\sin u \cdot u'$
$\tan u$	$\sec^2 u \cdot u'$	$\sec u$	$\sec u \tan u \cdot u'$
e^u	$e^u\,u'$	$\ln u$	$\dfrac{u'}{u}$
$a^u \;\; (a > 0)$	$a^u \ln a \cdot u'$	$\log_a u$	$\dfrac{u'}{u \ln a}$
$\ln \lvert u \rvert$	$\dfrac{u'}{u} \;\; (u \neq 0)$	$\sin^n x = (\sin x)^n$	$n \sin^{n-1} x \cos x$

IMPLICIT DIFFERENTIATION

Method

1. Differentiate both sides with respect to x.
2. Every term containing y gets a factor $\frac{dy}{dx}$ (chain rule).
3. Collect the $\frac{dy}{dx}$ terms on one side, factor, and solve.

Key pieces

$\frac{d}{dx}[y^n] = ny^{n-1}\frac{dy}{dx}$ $\qquad$ $\frac{d}{dx}[xy] = x\frac{dy}{dx} + y$

$\frac{d}{dx}[\sin y] = \cos y\,\frac{dy}{dx}$ $\qquad$ $\frac{d}{dx}[e^y] = e^y\frac{dy}{dx}$

Second derivative implicitly

Differentiate $\frac{dy}{dx}$ again with respect to x (quotient rule if needed), then substitute the expression for $\frac{dy}{dx}$ and simplify — often using the original equation.

Slope at a point

Substitute both x and y of the point into $\frac{dy}{dx}$. Check the point is on the curve first.

DERIVATIVES OF INVERSE FUNCTIONS

Formula

If $g = f^{-1}$, $f(a) = b$ and $f'(a) \neq 0$, then

$$g'(b) = \frac{1}{f'(a)} = \frac{1}{f'(g(b))}.$$

The slope of f^{-1} at (b, a) is the reciprocal of the slope of f at (a, b).

From a table of f and f'

To find $(f^{-1})'(b)$: locate the $x = a$ where $f(a) = b$, then take $1/f'(a)$ — **not** $1/f'(b)$.

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DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

$f(x)$	$f'(x)$	With chain rule	Result
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}\ \ (\lvert x\rvert<1)$	$\arcsin u$	$\dfrac{u'}{\sqrt{1-u^2}}$
$\arccos x$	$-\dfrac{1}{\sqrt{1-x^2}}\ \ (\lvert x\rvert<1)$	$\arccos u$	$-\dfrac{u'}{\sqrt{1-u^2}}$
$\arctan x$	$\dfrac{1}{1+x^2}$	$\arctan u$	$\dfrac{u'}{1+u^2}$

$\arcsin x = \sin^{-1} x$ *(not* $\frac{1}{\sin x}$*). Ranges:* arcsin: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; arccos: $[0, \pi]$; arctan: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

HIGHER-ORDER DERIVATIVES

$f''(x) = \dfrac{d^2y}{dx^2},\ f'''(x) = \dfrac{d^3y}{dx^3},\ f^{(n)}(x) = \dfrac{d^ny}{dx^n}$

$f^{(4)}$ is the fourth derivative, not f to the 4th power.

$\dfrac{d^n}{dx^n}[e^x] = e^x.$ $\sin x \to \cos x \to -\sin x \to -\cos x \to \sin x$ repeats every 4 derivatives.

WORKED EXAMPLES

Example. Differentiate $y = \sin(x^3)$ and $y = e^{\cos x}$.

$3x^2\cos(x^3);\ -\sin x\, e^{\cos x}$.

Example. Differentiate $y = \ln(x^2+1)$ and $y = 2^{3x}$.

$\dfrac{2x}{x^2+1};\ 2^{3x}\cdot\ln 2\cdot 3 = 3\ln 2\cdot 2^{3x}$.

Example. For $x^2+y^2=25$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $(3,4)$.

$2x+2yy'=0 \Rightarrow y' = -\dfrac{x}{y} = -\dfrac{3}{4}$.

$y'' = -\dfrac{y-xy'}{y^2} = -\dfrac{x^2+y^2}{y^3} = -\dfrac{25}{y^3} = -\dfrac{25}{64}$.

Example. For $xy+y^3=10$, find $\frac{dy}{dx}$ at $(9,1)$.

$y + xy' + 3y^2y' = 0 \Rightarrow y' = -\dfrac{y}{x+3y^2} = -\dfrac{1}{12}$.

Example. $f(x) = x^3 + x$ and $g = f^{-1}$. Find $g'(2)$.

$f(1)=2,\ f'(x)=3x^2+1 \Rightarrow f'(1)=4$. So $g'(2) = \dfrac{1}{4}$.

Example. Differentiate $y = \arctan(2x)$ and evaluate at $x = \frac{1}{2}$.

$y' = \dfrac{2}{1+4x^2}$; at $\frac{1}{2}$: $\dfrac{2}{2} = 1$.

COMMON ERRORS

- Forgetting the inner derivative: $\frac{d}{dx}\sin(3x) = 3\cos(3x)$, not $\cos(3x)$.
- Implicit: forgetting $\frac{dy}{dx}$ on a y-term, or the product rule on xy.
- Inverse functions: using $1/f'(b)$ instead of $1/f'(a)$ where $f(a)=b$.

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