AP® CALCULUS AB & BC — UNIT 4

Contextual Applications of Differentiation · Formula & Reference Sheet

THE DERIVATIVE IN CONTEXT

Interpreting $f'(a)$

"At $x = a$ [units of x], [quantity f] is changing at a rate of $f'(a)$ [units of f] per [unit of x]."

$f'(a) > 0$: increasing at a. $f'(a) < 0$: decreasing at a.

Units

f': (units of f) / (unit of x). f'': (units of f) / (unit of x)^2.

Example: $V(t)$ liters, t minutes: $V'(t)$ in liters per minute, $V''(t)$ in liters per minute per minute.

STRAIGHT-LINE MOTION

Quantity	Formula	Meaning
Position	$x(t)$ or $s(t)$	Location on the line at time t
Velocity	$v(t) = x'(t)$	$v > 0$: moving right (positive direction); $v < 0$: moving left
Acceleration	$a(t) = v'(t) = x''(t)$	Rate of change of velocity
Speed	$\lvert v(t) \rvert$	Always ≥ 0

Speeding up: $v(t)$ and $a(t)$ have the **same** sign.

Slowing down: $v(t)$ and $a(t)$ have **opposite** signs.

Changes direction at $t = c$ when $v(t)$ **changes sign** at c — $v(c) = 0$ alone is not enough.

At rest: $v(t) = 0$.

RATES OF CHANGE IN OTHER CONTEXTS

Any derivative is a rate: $P'(t)$ people/year, $C'(x)$ dollars per item (marginal cost — approximately the cost of one more item), $T'(t)$ degrees per minute. State the value, the units, and the time or input it refers to.

RELATED RATES

Method

1. Name the changing quantities; write the given rates and the rate wanted (with signs).
2. Write an equation relating the quantities.
3. Differentiate both sides with respect to t (every variable gets $\frac{d}{dt}$).
4. **Then** substitute the values at the instant asked about, and solve.

⚠ Substituting a changing value before differentiating turns its rate into 0.

Equation	Rate equation
Circle $A = \pi r^2$	$A' = 2\pi r\, r'$
Sphere $V = \frac{4}{3}\pi r^3$	$V' = 4\pi r^2\, r'$
Cylinder $V = \pi r^2 h$	$V' = \pi(2rh\, r' + r^2 h')$
Cone $V = \frac{1}{3}\pi r^2 h$	$V' = \frac{\pi}{3}(2rh\, r' + r^2 h')$
Right triangle $x^2 + y^2 = z^2$	$2x\, x' + 2y\, y' = 2z\, z'$

Primes mean $\frac{d}{dt}$. For a cone with fixed shape, use similar triangles to remove r or h before differentiating.

LOCAL LINEARITY AND LINEARIZATION

Linearization at $x = a$

$L(x) = f(a) + f'(a)(x - a)$, so $f(x) \approx L(x)$ for x near a.

Change: $\Delta f \approx f'(a)\, \Delta x$.

Over- or underestimate?

Concave up near a ($f'' > 0$): tangent line lies below the curve $\Rightarrow$ $L(x)$ **underestimates**.

Concave down near a ($f'' < 0$): tangent line lies above $\Rightarrow$ $L(x)$ **overestimates**.

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L'HÔPITAL'S RULE

If $\lim_{x\to a}\dfrac{f(x)}{g(x)}$ has the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, f and g are differentiable near a, $g'(x)\neq 0$ near a (except possibly at a), and $\lim_{x\to a}\dfrac{f'(x)}{g'(x)}$ exists (or is $\pm\infty$), then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

*Also valid for one-sided limits and $x\to\pm\infty$. Differentiate top and bottom **separately** — not the quotient rule. Confirm the form each time before reapplying; on the AP exam, show that both limits are 0 (or both infinite).*

WORKED EXAMPLES

Example. $x(t)=t^3-6t^2+9t,\ t\geq 0$. When is the particle speeding up?

$v=3t^2-12t+9=3(t-1)(t-3)$, $a=6t-12$. $v<0$ on $(1,3)$, $a<0$ on $(0,2)$. Same sign on $(1,2)$ and $(3,\infty)$: speeding up there.

Example. A sphere's volume grows at $100\ \text{cm}^3/\text{s}$. Find $\frac{dr}{dt}$ when $r=5$ cm.

$V'=4\pi r^2 r'$, so $100=4\pi(25)r'$ and $r'=\dfrac{1}{\pi}\approx 0.318\ \text{cm/s}$.

Example. A 10 ft ladder's base slides away at $2\ \text{ft/s}$. How fast does the top fall when the base is 6 ft from the wall?

$x^2+y^2=100$; at $x=6$, $y=8$.

$2xx'+2yy'=0\Rightarrow y'=-\dfrac{x}{y}x'=-\dfrac{6}{8}(2)=-1.5$. The top falls at $1.5\ \text{ft/s}$.

Example. Approximate $\sqrt{4.1}$ with a tangent line. Over or under?

$f(x)=\sqrt{x}$, $a=4$: $L(x)=2+\frac{1}{4}(x-4)$, $L(4.1)=2.025$. $f''(x)=-\frac{1}{4}x^{-3/2}<0$: concave down, so an **overestimate** ($\sqrt{4.1}\approx 2.02485$).

Example. $\displaystyle\lim_{x\to 0}\frac{e^x-1-x}{x^2}$

Form $\frac{0}{0}$: $\dfrac{e^x-1}{2x}$, still $\frac{0}{0}$: $\dfrac{e^x}{2}\to\dfrac{1}{2}$.

Example. $\displaystyle\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}$

Form $\frac{\infty}{\infty}$: $\dfrac{1/x}{1/(2\sqrt{x})}=\dfrac{2}{\sqrt{x}}\to 0$.

COMMON ERRORS

- Saying the particle changes direction wherever $v=0$ without checking a sign change.
- Related rates: plugging in values before differentiating; dropping the sign of a decreasing quantity.
- Using L'Hôpital on a limit that is not $\frac{0}{0}$ or $\frac{\infty}{\infty}$, or using the quotient rule instead.

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