AP® CALCULUS AB & BC — UNIT 5

Analytical Applications of Differentiation · Formula & Reference Sheet

MEAN VALUE THEOREM (MVT)

If f is **continuous on** $[a, b]$ and **differentiable on** (a, b), then there is at least one c in (a, b) with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Some tangent line is parallel to the secant line. Motion: at some instant, velocity equals average velocity.

Special case $f(a) = f(b)$ (Rolle's theorem): $f'(c) = 0$ for some c in (a, b).

To use either theorem, state both hypotheses. To find c: set $f'(c)$ equal to the average rate and keep only solutions inside (a, b).

EXTREMA: DEFINITIONS AND THE EXTREME VALUE THEOREM

Definitions

Absolute max at c: $f(c) \geq f(x)$ for every x in the domain (absolute min: $\leq$).

Local (relative) max at an interior point c: $f(c) \geq f(x)$ for all x near c (local min: $\leq$).

On this sheet, local extrema are at interior points; endpoints are checked only as candidates for absolute extrema.

Critical point

An interior c in the domain of f where $f'(c) = 0$ or $f'(c)$ does not exist.

Every local extremum occurs at a critical point, but not every critical point is an extremum ($f(x) = x^3$ at 0).

Extreme Value Theorem

If f is continuous on the closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.

INCREASING / DECREASING AND THE FIRST DERIVATIVE TEST

$f' > 0$ on an interval $\Rightarrow$ f increasing there.

$f' < 0$ on an interval $\Rightarrow$ f decreasing there.

Sign chart: find the critical points, test the sign of f' on each interval between them.

f' at a critical point c	Conclusion
changes $+$ to $-$	local max at c
changes $-$ to $+$	local min at c
does not change sign	no local extremum

Requires f continuous at c. (A jump at c can defeat the test.)

ABSOLUTE EXTREMA ON $[a, b]$ — CANDIDATES TEST

For f continuous on $[a, b]$: evaluate f at every critical point in (a, b) **and** at $x = a$ and $x = b$. The largest value is the absolute max; the smallest is the absolute min.

f continuous on any interval (open or not) with exactly one critical point there: if that point is a local max (min), it is the absolute max (min) on the interval.

CONCAVITY, INFLECTION POINTS, SECOND DERIVATIVE TEST

Concavity

Concave up on an interval $\iff$ f' is increasing there; $f'' > 0$ on the interval guarantees it.

Concave down $\iff$ f' decreasing; $f'' < 0$ guarantees it.

Point of inflection

A point where f is continuous and the **concavity changes**. Candidates: $f''(c) = 0$ or $f''(c)$ undefined — then confirm f'' changes sign.

Second Derivative Test (at c with $f'(c) = 0$)

$f''(c) > 0$	local min at c
$f''(c) < 0$	local max at c
$f''(c) = 0$ or undefined	inconclusive — use the First Derivative Test

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$f''(c) = 0$ alone is not enough: x^4 has $f''(0) = 0$ and no inflection point. Inflection also occurs where f'' is undefined, e.g. $x^{1/3}$ at 0.

Reading f from the graph of f'

f increasing where f' is above the axis, decreasing where below · local extremum where f' crosses the axis · concave up where f' is increasing, concave down where f' is decreasing · inflection point where f' has a local max or min.

OPTIMIZATION

1. Write the quantity to optimize; use the constraint to reduce it to one variable.
2. State the domain from the context.
3. Find critical points; justify the max/min (Candidates Test on a closed interval, or First/Second Derivative Test).
4. Answer the question asked, with units.

Classic setup	Result
Rectangle, fixed perimeter P, max area	square, side $P/4$
Rectangle, fixed area A, min perimeter	square, side $\sqrt{A}$
Open box from an $s \times s$ sheet, cut $x \times x$ corners, max volume	$x = s/6$

BEHAVIOR OF IMPLICIT RELATIONS

From $\frac{dy}{dx} = \frac{N}{D}$ (found implicitly): **horizontal tangent** where $N = 0$ and $D \neq 0$; **vertical tangent** where $D = 0$ and $N \neq 0$. Solve together with the original equation so the point is on the curve. Use $\frac{d^2y}{dx^2}$ at the point for concavity.

WORKED EXAMPLES

Example. Find the c given by the MVT for $f(x) = x^2$ on $[1, 3]$.

Polynomial: continuous and differentiable. Average rate $\frac{9-1}{3-1} = 4$; $2c = 4 \Rightarrow c = 2$, which is in $(1, 3)$.

Example. Absolute extrema of $f(x) = x^3 - 3x$ on $[0, 2]$.

$f' = 3x^2 - 3 = 0 \Rightarrow x = 1$ (only $x = 1$ is in the interval). $f(0) = 0$, $f(1) = -2$, $f(2) = 2$. Max 2 at $x = 2$; min -2 at $x = 1$.

Example. Classify the critical points and inflection points of $f(x) = x^4 - 4x^3$.

$f' = 4x^2(x - 3)$: $f' < 0$ on $(-\infty, 0)$ and $(0, 3)$, $f' > 0$ on $(3, \infty)$ — no extremum at 0, local min at 3.
$f'' = 12x(x - 2)$ changes sign at 0 and 2: inflection points at $x = 0$ and $x = 2$.

Example. Open box from a 12×12 sheet: maximize V.

$V = x(12 - 2x)^2,\ 0 < x < 6$.
$V' = (12 - 2x)(12 - 6x) = 0 \Rightarrow x = 2$ ($x = 6$ is excluded). V' changes $+$ to $-$ at 2: max $V(2) = 128$.

Example. Horizontal tangents of $x^2 + xy + y^2 = 3$.

$\frac{dy}{dx} = -\frac{2x+y}{x+2y}$. Numerator $0 \Rightarrow y = -2x$; substitute: $3x^2 = 3 \Rightarrow x = \pm 1$. Points $(1, -2)$ and $(-1, 2)$; denominator $\mp 3 \neq 0$.

COMMON ERRORS

- Forgetting the endpoints in the Candidates Test, or including them as local extrema.
- Calling every point with $f'' = 0$ an inflection point without a sign change.
- Using the MVT without stating continuity on $[a, b]$ and differentiability on (a, b).

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