AP® CALCULUS AB & BC — UNIT 6

Integration and Accumulation of Change · Formula & Reference Sheet

Sections marked **BC ONLY** are not on the AB exam.

ACCUMULATION OF CHANGE

If $r(t)$ is a rate of change, $\displaystyle\int_a^b r(t)\,dt$ is the **net change** in the quantity from $t = a$ to $t = b$, in (rate units) $\times$ (time units). Area above the t-axis counts positive; area below counts negative.

RIEMANN SUMS AND THE TRAPEZOIDAL RULE

Equal subintervals: $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\,\Delta x$.

Left	$L_n = \Delta x\,[f(x_0) + f(x_1) + \cdots + f(x_{n-1})]$
Right	$R_n = \Delta x\,[f(x_1) + f(x_2) + \cdots + f(x_n)]$
Midpoint	$M_n = \Delta x\,\left[f\left(\frac{x_0+x_1}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right]$
Trapezoidal	$T_n = \dfrac{\Delta x}{2}\,[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)] = \dfrac{L_n + R_n}{2}$
Unequal widths	Use each subinterval's own width: $\sum f(\text{sample point}) \cdot (x_i - x_{i-1})$; trapezoid: $\sum \frac{f(x_{i-1})+f(x_i)}{2}(x_i - x_{i-1})$.

Over- or underestimate? Monotonicity decides left/right sums; concavity decides midpoint/trapezoid.

f on $[a, b]$	Left	Right	Midpoint	Trapezoid
increasing	under	over	— (use concavity)	— (use concavity)
decreasing	over	under	— (use concavity)	— (use concavity)
concave up	— (use monotonicity)	— (use monotonicity)	under	over
concave down	— (use monotonicity)	— (use monotonicity)	over	under

For a linear f, midpoint and trapezoid sums are exact; left and right sums are exact only when f is constant. The definite integral is the limit of the sums: $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$ *(for f continuous).*

FUNDAMENTAL THEOREM OF CALCULUS

Accumulation functions

If f is continuous and $g(x) = \displaystyle\int_a^x f(t)\,dt$, then $g'(x) = f(x)$.

Variable upper limit: $\dfrac{d}{dx}\displaystyle\int_a^{u(x)} f(t)\,dt = f(u(x))\,u'(x)$.

Variable lower limit: $\dfrac{d}{dx}\displaystyle\int_x^a f(t)\,dt = -f(x)$.

Both variable: $\dfrac{d}{dx}\displaystyle\int_{v(x)}^{u(x)} f(t)\,dt = f(u)\,u' - f(v)\,v'$.

Evaluating definite integrals

If $F' = f$ and f is continuous on $[a, b]$:

$\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$.

Net change: $\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$, so

$f(b) = f(a) + \displaystyle\int_a^b f'(x)\,dx$.

Behavior of $g(x) = \int_a^x f(t)\,dt$

g increasing where $f > 0$; decreasing where $f < 0$; local extremum where f changes sign; $g'' = f'$, so g is concave up where f is increasing; inflection where f changes from increasing to decreasing (or vice versa).

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PROPERTIES OF DEFINITE INTEGRALS

$\int_a^a f(x)\,dx = 0$	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	$\int_a^b [f \pm g]\,dx = \int_a^b f\,dx \pm \int_a^b g\,dx$
$\int_a^b f\,dx + \int_b^c f\,dx = \int_a^c f\,dx$	f odd: $\int_{-a}^a f\,dx = 0$; f even: $\int_{-a}^a f\,dx = 2\int_0^a f\,dx$

Geometry helps: integrals of lines and semicircles are areas of triangles, rectangles and circles (sign by side of the axis).

ANTIDERIVATIVES

$\int$	Result	$\int$	Result
$x^n\,dx\ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1} + C$	$\dfrac{1}{x}\,dx$	$\ln\lvert x\rvert + C$
$e^x\,dx$	$e^x + C$	$a^x\,dx\ (a > 0,\ a \neq 1)$	$\dfrac{a^x}{\ln a} + C$
$\sin x\,dx$	$-\cos x + C$	$\cos x\,dx$	$\sin x + C$
$\sec^2 x\,dx$	$\tan x + C$	$\csc^2 x\,dx$	$-\cot x + C$
$\sec x \tan x\,dx$	$\sec x + C$	$\csc x \cot x\,dx$	$-\csc x + C$
$\dfrac{dx}{\sqrt{1-x^2}}$	$\arcsin x + C$	$\dfrac{dx}{1+x^2}$	$\arctan x + C$
$\dfrac{dx}{\sqrt{a^2-x^2}}\ (a > 0)$	$\arcsin\dfrac{x}{a} + C$	$\dfrac{dx}{a^2+x^2}\ (a \neq 0)$	$\dfrac{1}{a}\arctan\dfrac{x}{a} + C$
$\tan x\,dx$	$-\ln\lvert\cos x\rvert + C$	$\cot x\,dx$	$\ln\lvert\sin x\rvert + C$

SUBSTITUTION

$\displaystyle\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$, with $u = g(x)$, $du = g'(x)\,dx$.

Definite integrals: change the limits too, $\displaystyle\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$ — or substitute back before evaluating.

Long division first when degree of numerator $\geq$ degree of denominator.

Completing the square:
$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)$, then an $\arctan$ or $\arcsin$ form.

INTEGRATION BY PARTS BC ONLY

$$\int u\,dv = uv - \int v\,du \qquad \int_a^b u\,dv = \Big[uv\Big]_a^b - \int_a^b v\,du$$

Choose u to get simpler when differentiated: logarithms, inverse trig, then powers of x, then trig and exponentials.

$\int xe^x\,dx = (x-1)e^x + C$

$\int \ln x\,dx = x\ln x - x + C$

$\int x\cos x\,dx = x\sin x + \cos x + C$

PARTIAL FRACTIONS — DISTINCT LINEAR FACTORS BC ONLY

$\dfrac{px+q}{(x-a)(x-b)} = \dfrac{A}{x-a} + \dfrac{B}{x-b}\ (a \neq b)$. Multiply through by $(x-a)(x-b)$ and set $x = a$, then $x = b$. Each term integrates to $A\ln\lvert x-a\rvert + B\ln\lvert x-b\rvert + C$.

IMPROPER INTEGRALS BC ONLY

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

Unbounded at $x = b$: $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$

Converges if the limit is a finite number; otherwise diverges. An interior discontinuity: split there; both pieces must converge.

$\displaystyle\int_1^\infty \frac{dx}{x^p}$ converges (to $\frac{1}{p-1}$) if $p > 1$; diverges if $p \leq 1$.

$\displaystyle\int_0^1 \frac{dx}{x^p}$ converges (to $\frac{1}{1-p}$) if $p < 1$; diverges if $p \geq 1$.

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WORKED EXAMPLES

Example. $f(x) = x^2$ **on** $[0, 2]$**,** $n = 4$**: left, right, midpoint, trapezoid.**
$\Delta x = 0.5$. $L_4 = 1.75$, $R_4 = 3.75$, $M_4 = 2.625$, $T_4 = 2.75$. Exact $\frac{8}{3} \approx 2.667$. Increasing: L under, R over; concave up: M under, T over.

Example. $\dfrac{d}{dx}\displaystyle\int_0^{x^3} \cos t\, dt$ **and** $\dfrac{d}{dx}\displaystyle\int_x^5 \sqrt{1+t^2}\, dt$
Upper limit x^3: $\cos(x^3) \cdot 3x^2$. Lower limit x: $-\sqrt{1+x^2}$.

Example. $\displaystyle\int_0^1 2x(x^2+1)^3\, dx$
$u = x^2 + 1$, $du = 2x\, dx$, u from 1 to 2:
$\displaystyle\int_1^2 u^3\, du = \frac{16-1}{4} = \frac{15}{4}$.

Example. $\displaystyle\int \frac{dx}{x^2+4x+5}$ **and** $\displaystyle\int \frac{x^2+1}{x+1}\, dx$
$x^2 + 4x + 5 = (x+2)^2 + 1 \Rightarrow \arctan(x+2) + C$.
Division: $x - 1 + \frac{2}{x+1} \Rightarrow \frac{x^2}{2} - x + 2\ln|x+1| + C$.

Example. BC $\displaystyle\int_0^1 xe^x\, dx$
$u = x,\ dv = e^x dx$: $\left[(x-1)e^x\right]_0^1 = 0 - (-1) = 1$.

Example. BC $\displaystyle\int \frac{dx}{(x-1)(x+2)}$
$1 = A(x+2) + B(x-1)$: $x = 1 \Rightarrow A = \frac{1}{3}$; $x = -2 \Rightarrow B = -\frac{1}{3}$. Result $\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C$.

Example. BC $\displaystyle\int_1^\infty \frac{dx}{x^2}$ **and** $\displaystyle\int_0^\infty e^{-x}\, dx$
$\lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1$; $\lim_{b\to\infty}\left(1 - e^{-b}\right) = 1$. Both converge to 1.

COMMON ERRORS

- Using monotonicity to judge a trapezoid or midpoint sum (that needs concavity), or vice versa.
- Forgetting the chain-rule factor $u'(x)$ when the upper limit is a function of x.
- Changing variables with u-substitution but keeping the old x-limits.
- $\int \frac{1}{x}\, dx = \ln|x| + C$ — the absolute value matters.

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