AP® CALCULUS AB & BC — UNIT 7

Differential Equations · Formula & Reference Sheet

Sections marked **BC ONLY** are not on the AB exam.

MODELING AND VERIFYING SOLUTIONS

Translating words

"Rate of change of y is proportional to y": $\dfrac{dy}{dt} = ky$.

"...proportional to the difference between y and M": $\dfrac{dy}{dt} = k(y - M)$.

Verifying a solution

Differentiate the proposed y, substitute y and its derivatives into the equation, and confirm both sides agree for all x in the interval. For a particular solution, also check the initial condition.

Second derivative from a DE: differentiate $\frac{dy}{dx} = f(x, y)$ implicitly, then substitute $\frac{dy}{dx}$.

SLOPE FIELDS

At each point (x, y), draw a short segment with slope $\frac{dy}{dx}$ evaluated there. A solution curve is tangent to every segment it passes through.

- $\frac{dy}{dx} = f(x)$ only: slopes are the same along each **vertical** line; solutions are vertical shifts of each other.
- $\frac{dy}{dx} = g(y)$ only: slopes are the same along each **horizontal** line.
- Matching a field: check where slopes are 0, positive, negative, and test a few points.

Equilibrium solutions

A constant function $y = c$ is a solution exactly when $\frac{dy}{dx} = 0$ at **every** point of the line $y = c$.

$\frac{dy}{dx} = xy$: $y = 0$ is an equilibrium solution. The line $x = 0$ also has zero slopes, but it is vertical — it is not the graph of a solution.

$\frac{dy}{dx} = x + y$: slopes are 0 along $y = -x$, but $y = -x$ is **not** a solution (there $y' = -1 \neq 0$). The straight-line solution is $y = -x - 1$.

EULER'S METHOD BC ONLY

For $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, step size Δx:

$x_{n+1} = x_n + \Delta x, \qquad y_{n+1} = y_n + f(x_n, y_n)\,\Delta x$

Each step follows the tangent line from the current point.

Solution concave up on the steps ⇒ Euler **underestimates**. Concave down ⇒ Euler **overestimates**.

Use $\frac{d^2y}{dx^2}$ from the DE to decide concavity.

SEPARATION OF VARIABLES

1. Write $\frac{dy}{dx} = g(x)\,h(y)$ as $\dfrac{1}{h(y)}\,dy = g(x)\,dx$.
2. Integrate both sides; add **one** constant C.
3. Use the initial condition to find C right away.
4. Solve for y; choose the sign or branch that fits the initial condition.

Dividing by $h(y)$ can lose constant solutions where $h(y) = 0$; check them separately.

A particular solution is valid on an interval containing x_0 on which it is defined and differentiable.

EXPONENTIAL MODELS

$\dfrac{dy}{dt} = ky \implies y = y_0 e^{kt}$

$k > 0$: growth; doubling time $\dfrac{\ln 2}{k}$.

$k < 0$: decay; half-life $\dfrac{\ln 2}{|k|}$.

$\dfrac{dy}{dt} = k(y - M) \implies y = M + (y_0 - M)e^{kt}$

Newton's law of cooling (M = surrounding temperature, $k < 0$): $y \to M$.

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LOGISTIC MODELS BC ONLY

$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$, $k > 0$ — carrying capacity L.

Solution: $P(t) = \frac{L}{1 + Ae^{-kt}}$, with $A = \frac{L - P_0}{P_0}$.

- If $P_0 > 0$: $\lim_{t\to\infty} P(t) = L$.
- P grows fastest when $P = \frac{L}{2}$ (inflection point of $P(t)$).
- Equilibrium solutions: $P = 0$ and $P = L$.

WORKED EXAMPLES

Example. $\frac{dy}{dx} = xy$, $y(0) = 2$.

$\int \frac{dy}{y} = \int x\,dx \Rightarrow \ln|y| = \frac{x^2}{2} + C$. $y(0) = 2 \Rightarrow C = \ln 2$. $y = 2e^{x^2/2}$.

Example. $\frac{dy}{dx} = \frac{x}{y}$, $y(0) = -3$.

$\int y\,dy = \int x\,dx \Rightarrow y^2 = x^2 + C$, $C = 9$. Since $y(0) < 0$: $y = -\sqrt{x^2 + 9}$.

Example. A quantity decays exponentially and halves every 5 years. Find k and the time to reach 10%.

$k = -\frac{\ln 2}{5} \approx -0.1386$. $0.1 = e^{kt} \Rightarrow t = \frac{\ln 10}{\ln 2} \cdot 5 \approx 16.61$ years.

Example. BC **Euler:** $\frac{dy}{dx} = x + y$, $y(0) = 1$**, two steps of** 0.1.

$y_1 = 1 + 0.1(0 + 1) = 1.1$;

$y_2 = 1.1 + 0.1(0.1 + 1.1) = 1.22$. Exact $y = 2e^x - x - 1$ gives $y(0.2) \approx 1.2428$; $y'' = 1 + x + y > 0$, so Euler underestimates.

Example. BC $\frac{dP}{dt} = 0.2P\left(1 - \frac{P}{500}\right)$, $P(0) = 100$.

$\lim_{t\to\infty} P = 500$; fastest growth at $P = 250$, where $\frac{dP}{dt} = 0.2(250)(0.5) = 25$. $P(t) = \frac{500}{1 + 4e^{-0.2t}}$.

Example. $\frac{dy}{dx} = x + y$**: find** $\frac{d^2y}{dx^2}$ **in terms of** x **and** y.

$\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx} = 1 + x + y$.

COMMON ERRORS

- Adding $+C$ after solving for y instead of right after integrating.
- Calling a line of zero slopes an equilibrium solution when it is not the graph of a solution.
- Choosing the wrong sign after taking a square root — the initial condition decides.

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