DREAMMONGERI EDUCATION

AP® CALCULUS AB & BC — UNIT 8

Applications of Integration · Formula & Reference Sheet

Sections marked **BC ONLY** are not on the AB exam.

AVERAGE VALUE OF A FUNCTION

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

The height of the rectangle on $[a,b]$ with the same signed area as f.

Mean Value Theorem for integrals: if f is continuous on $[a,b]$, then $f(c) = f_{\text{avg}}$ for some c in $[a,b]$.

⚠ Average value of f is not the average rate of change $\frac{f(b)-f(a)}{b-a}$.

POSITION, VELOCITY, ACCELERATION

Velocity from acceleration	$v(t) = v(t_0) + \int_{t_0}^{t} a(s)\,ds$
Position from velocity	$x(t) = x(t_0) + \int_{t_0}^{t} v(s)\,ds$
Displacement on $[a,b]$	$\int_a^b v(t)\,dt$ (signed)
Total distance on $[a,b]$	$\int_a^b \lvert v(t)\rvert\,dt$ (split where v changes sign, or use a calculator)

ACCUMULATION IN CONTEXT

Amount at time t = initial amount + $\int_0^t \big(\text{rate in} - \text{rate out}\big)\,ds$. The amount is increasing when rate in $>$ rate out. Absolute max/min of the amount on $[a,b]$: Candidates Test at the endpoints and where rate in $=$ rate out.

AREA BETWEEN CURVES

With respect to x (vertical slices)

$$A = \int_a^b \big[\,\text{top} - \text{bottom}\,\big]\,dx$$

With respect to y (horizontal slices)

$A = \int_c^d \big[\,\text{right} - \text{left}\,\big]\,dy$, curves written as $x = g(y)$, limits are y-values.

When the curves cross

Find every intersection, split the integral there, and use top − bottom on each piece. Equivalently $\int_a^b \lvert f(x) - g(x)\rvert\,dx$.

Choose dy when the region is bounded left and right by single curves, or when dx needs several pieces.

VOLUMES WITH KNOWN CROSS SECTIONS

$V = \int_a^b A(x)\,dx$ for cross sections perpendicular to the x-axis (use $A(y)\,dy$ for the y-axis). s = length of the slice across the base region.

Cross section	Area A
Square with side s	s^2
Rectangle with base s, height h	sh
Equilateral triangle with side s	$\frac{\sqrt{3}}{4}s^2$
Isosceles right triangle, leg s	$\frac{1}{2}s^2$
Isosceles right triangle, hypotenuse s	$\frac{1}{4}s^2$
Semicircle with diameter s	$\frac{\pi}{8}s^2$

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VOLUMES OF REVOLUTION — DISC AND WASHER

Disc (region touches the axis)

$$V = \pi \int_a^b [R(x)]^2 \, dx$$

Washer (gap between region and axis)

$$V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2 \right) dx$$

⚠ $R^2 - r^2$, not $(R - r)^2$.

Radii are distances to the axis

Horizontal axis $y = k$: R or $r = |f(x) - k|$.

Vertical axis $x = h$: integrate in y; R or $r = |g(y) - h|$.

R = distance to the far boundary, r = distance to the near boundary.

ARC LENGTH BC ONLY

For $y = f(x)$ with f' continuous on $[a, b]$: $L = \int_a^b \sqrt{1 + \left(f'(x)\right)^2} \, dx$. For $x = g(y)$: $L = \int_c^d \sqrt{1 + \left(g'(y)\right)^2} \, dy$.

Arc length of parametric curves and distance traveled along them: see Unit 9.

WORKED EXAMPLES

Example. Average value of $f(x) = x^2$ on $[0, 3]$.

$\frac{1}{3} \int_0^3 x^2 \, dx = \frac{1}{3} \cdot 9 = 3$.

Example. $v(t) = t^2 - 4$ on $[0, 3]$: displacement and total distance.

Displacement $\int_0^3 (t^2 - 4) \, dt = 9 - 12 = -3$. v changes sign at $t = 2$: distance $\int_0^2 (4 - t^2) \, dt + \int_2^3 (t^2 - 4) \, dt = \frac{16}{3} + \frac{7}{3} = \frac{23}{3}$.

Example. Area between $y = x$ and $y = x^2$.

They meet at $x = 0, 1$; $x \geq x^2$ there: $\int_0^1 (x - x^2) \, dx = \frac{1}{6}$.

Example. Area between $x = y^2$ and $x = y + 2$.

$y^2 = y + 2 \Rightarrow y = -1, 2$; right curve $x = y + 2$: $\int_{-1}^2 (y + 2 - y^2) \, dy = \frac{9}{2}$.

Example. Region under $y = \sqrt{x}$, $0 \leq x \leq 4$, rotated about the x-axis.

Disc: $\pi \int_0^4 x \, dx = 8\pi$.

Example. Region between $y = x$ and $y = x^2$ rotated about $y = 0$, then about $y = -1$.

About $y = 0$: $\pi \int_0^1 (x^2 - x^4) \, dx = \frac{2\pi}{15}$. About $y = -1$: $R = x + 1$, $r = x^2 + 1$: $\pi \int_0^1 \left[(x + 1)^2 - (x^2 + 1)^2 \right] dx = \frac{7\pi}{15}$.

Example. Same base region; square cross sections perpendicular to the x-axis.

$s = x - x^2$: $\int_0^1 (x - x^2)^2 \, dx = \frac{1}{30}$.

Example. BC Arc length of $y = \frac{2}{3} x^{3/2}$, $0 \leq x \leq 3$.

$y' = x^{1/2}$: $\int_0^3 \sqrt{1 + x} \, dx = \frac{2}{3} \left(8 - 1 \right) = \frac{14}{3}$.

COMMON ERRORS

- Using $\int v \, dt$ when the question asks for total distance.
- Washer: squaring $R - r$ instead of subtracting $R^2 - r^2$; radius measured from the wrong axis.
- Mixing variables: an integral in dy needs y-limits and curves written as $x = g(y)$.

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