AP® CALCULUS BC — UNIT 9

Parametric Equations, Polar Coordinates, and Vector-Valued Functions · Formula & Reference Sheet

BC ONLY This whole unit is tested on the BC exam only.

PARAMETRIC EQUATIONS: DERIVATIVES

Curve $x = x(t)$, $y = y(t)$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \left(\frac{dx}{dt} \neq 0\right)$$

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt}$$

⚠ $\dfrac{d^2y}{dx^2} \neq \dfrac{d^2y/dt^2}{d^2x/dt^2}$.

Horizontal tangent: $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$.

Vertical tangent: $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.

If both are 0 at t_0, neither test applies; examine $\frac{dy}{dx}$ as $t \to t_0$.

Tangent line at t_0: through $(x(t_0), y(t_0))$ with slope $\left.\frac{dy}{dx}\right|_{t_0}$.

ARC LENGTH AND DISTANCE TRAVELED

$L = \displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ — the length of the path traced for $a \le t \le b$, which is the distance traveled by a particle at $(x(t), y(t))$. If the curve retraces itself, this counts the retraced part again.

VECTOR-VALUED FUNCTIONS AND MOTION IN THE PLANE

Quantity	Formula	Notes
Position	$\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$	Point on the path at time t
Velocity	$\mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t),\, y'(t)\rangle$	Tangent to the path
Acceleration	$\mathbf{a}(t) = \mathbf{v}'(t) = \langle x''(t),\, y''(t)\rangle$	Derivative of velocity
Speed	$\lvert\mathbf{v}(t)\rvert = \sqrt{(x'(t))^2 + (y'(t))^2}$	A scalar, ≥ 0
Position from velocity	$\mathbf{r}(t) = \mathbf{r}(t_0) + \displaystyle\int_{t_0}^{t} \mathbf{v}(s)\, ds$	Integrate each component
Displacement	$\displaystyle\int_a^b \mathbf{v}(t)\, dt = \mathbf{r}(b) - \mathbf{r}(a)$	A vector
Distance traveled	$\displaystyle\int_a^b \lvert\mathbf{v}(t)\rvert\, dt$	Integral of speed

Moving right when $x'(t) > 0$; moving up when $y'(t) > 0$. At rest when $x'(t) = y'(t) = 0$.

Speed is increasing at t when $\dfrac{d}{dt}\lvert\mathbf{v}(t)\rvert > 0$.

POLAR COORDINATES

Conversion

$x = r\cos\theta,\ y = r\sin\theta,\ r^2 = x^2 + y^2,\ \tan\theta = \dfrac{y}{x}\ (x \neq 0)$

Slope of $r = f(\theta)$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

Horizontal tangent: $\frac{dy}{d\theta} = 0,\ \frac{dx}{d\theta} \neq 0$. Vertical: $\frac{dx}{d\theta} = 0,\ \frac{dy}{d\theta} \neq 0$.

Rate of change of r

$\frac{dr}{d\theta} > 0$ with $r > 0$: the curve is moving away from the origin;
$\frac{dr}{d\theta} < 0$ with $r > 0$: moving toward it.

Symmetry (each condition is sufficient, not necessary)

About the x-axis: $f(-\theta) = f(\theta)$.

About the y-axis: $f(\pi - \theta) = f(\theta)$.

About the origin: $f(\theta + \pi) = f(\theta)$.

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POLAR AREA

One curve: $A = \dfrac{1}{2}\displaystyle\int_{\alpha}^{\beta} r^2\,d\theta$

Between curves (R outer, r inner): $A = \dfrac{1}{2}\displaystyle\int_{\alpha}^{\beta} \left(R^2 - r^2\right) d\theta$

⚠ $R^2 - r^2$, not $(R - r)^2$.

Find α, β from intersections ($f(\theta) = g(\theta)$) or where $r = 0$. Curves may also meet at the origin at different θ.

Useful: $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$.

WORKED EXAMPLES

Example. $x = t^2,\ y = t^3$**: $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $t = 1$.**

$\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2}$ $(t \neq 0)$; $\frac{d^2y}{dx^2} = \frac{3/2}{2t} = \frac{3}{4t}$. At $t = 1$: $\frac{3}{2}$ and $\frac{3}{4}$.

Example. Length of $x = t^2,\ y = t^3$, $0 \le t \le 2$.

$\int_0^2 \sqrt{4t^2 + 9t^4}\,dt = \int_0^2 t\sqrt{4 + 9t^2}\,dt$
$= \frac{1}{27}\left(40^{3/2} - 8\right) \approx 9.073.$

Example. $\mathbf{v}(t) = \langle 2t,\ e^t\rangle$, $\mathbf{r}(0) = \langle 1, 0\rangle$. Find $\mathbf{r}(t)$ and the speed at $t = 0$.

$\mathbf{r}(t) = \langle t^2 + 1,\ e^t - 1\rangle$. Speed $\sqrt{0^2 + 1^2} = 1$.

Example. Slope of $r = \theta$ at $\theta = \frac{\pi}{2}$.

$\frac{dy}{dx} = \frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta} = \frac{1+0}{0-\pi/2} = -\frac{2}{\pi}$.

Example. Area inside $r = 2\cos\theta$.

Traced once for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$: $\frac{1}{2}\int 4\cos^2\theta\,d\theta = \pi$ (a circle of radius 1).

Example. Area of one petal of $r = \sin(2\theta)$.

Petal for $0 \le \theta \le \frac{\pi}{2}$: $\frac{1}{2}\int_0^{\pi/2} \sin^2(2\theta)\,d\theta = \frac{\pi}{8}$.

Example. Area inside $r = 1 + \cos\theta$ and outside $r = 1$.

Intersect at $\theta = \pm\frac{\pi}{2}$:
$\frac{1}{2}\int_{-\pi/2}^{\pi/2} \left[(1+\cos\theta)^2 - 1\right] d\theta = 2 + \frac{\pi}{4}$.

COMMON ERRORS

- Dividing $\frac{d^2y}{dt^2}$ by $\frac{d^2x}{dt^2}$ for the parametric second derivative.
- Reporting speed as a vector, or displacement as a number when a vector is asked for.
- Polar area: forgetting the $\frac{1}{2}$, or using θ-limits that trace part of the curve twice.

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